Chapter 14

GAUSS'S LAW

A.) Flux:

1.) The word flux denotes a passage of something through a boundary or across a border (an influx of immigrants means immigrants are passing over a country's borders).

The concept has mathematical implications. To understand it, an example is in order.

2.) Assume we have a hot, irregularly shaped object. Assume also that there exists a vector h that defines the direction and magnitude of the heat flow per unit area per unit time in the space around the object. To determine how much heat leaves the object per unit time, we could use the following approach:

a.) Define a closed surface S around the object (the surface can be any shape--it doesn't have to parallel the contour of the hot object).

b.) On the surface S, define a differential area dS.

c.) Define a vector normal (i.e., perpendicular) to the surface dS

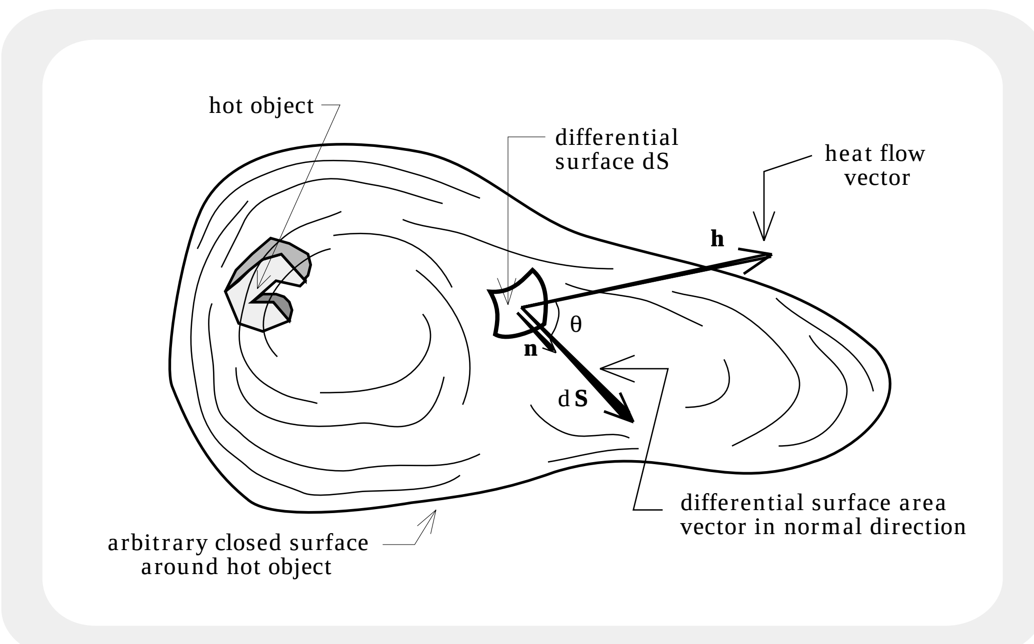


FIGURE 14.1

and pointed outward. Call this normal vector n.

d.) Define a differential surface area vector dS whose magnitude equals the differential surface area dS and whose direction is perpendicular to the surface's face (i.e., in the direction of n).

Mathematically, this vector will equal:

$$\mathbf{dS} = dS\mathbf{n}.$$

e.) Figure 14.1 on the previous page shows all of our defined quantities: the hot object, the arbitrary surface S, the normal vector n, the differential surface dS, the differential surface area vector dS, and the heat flow vector h.

f.) The heat flow from the object per unit time will equal the heat flow through the surface S per unit time. This is called the heat flux Φ_h (per unit time) through S.

The heat flux (per unit time) can be deduced by deriving a general expression for the heat flow through the differential surface dS, and then by integrating that expression over the entire surface.

g.) By the definition of a dot product, $\mathbf{h} \cdot d\mathbf{S}$ gives us the component of h in the direction of dS (the component of h perpendicular to dS produces no flux through the surface), or the amount of heat that passes through dS (per unit time). The units are:

(heat flow/unit area/unit time)(unit area) = heat flow/unit time.

h.) Integrating the dot product gives us the total heat flux (per unit time) through the entire surface. Mathematically, this is written:

$$\Phi_h = \int_S \mathbf{h} \bullet d\mathbf{S}.$$

Note: This integral does not show limits as usual. Instead, the letter "S" is used to show that the integral is a summation over a surface. In other words, the actual limits will depend upon the geometry of the surface itself (you will see why this isn't a problem shortly). Note also that the capital Greek letter phi is used to denote flux, and that that symbol is subscripted to denote what kind of flux it is. In this case, it is a heat flux.

3.) We have examined a situation in which there is a vector-defined quantity (heat flow) that has passed across a boundary, hence creating a flux. What is important is that any vector quantity can have a flux associated with it.

B.) Gauss's Law in General:

1.) Sometime around the turn of the 1800's, Carl Friedrich Gauss made some interesting observations concerning electric fields. Specifically, he suggested that the electric flux generated by an electric field passing through a closed surface S must be related solely to the charge enclosed within the surface. His reasoning follows:

a.) If an imaginary, closed surface is defined in the vicinity of an electric field and there is no net charge inside the surface, the electric field lines entering the surface will exit at some point and the net electric flux through the surface will be zero (see Figure 14.2).

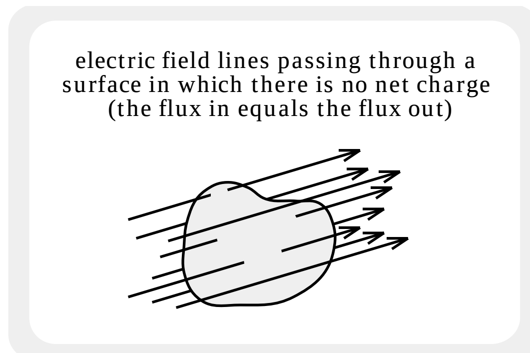


FIGURE 14.2

b.) If, on the other hand, there is a charge inside the surface, then an excess of electric field lines will either enter or exit the surface (depending upon whether the net charge is positive or negative--remember, electric field lines leave positive charges, enter negative charges) and the electric flux will not be zero (see Figure 14.3).

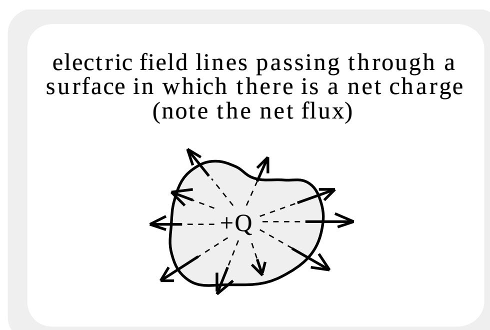


FIGURE 14.3

c.) As the electric flux through the surface is dependent upon the charge inside the surface, the electric flux must be proportional to that enclosed charge. After determining the proportionality constant, Gauss was able to write the relationship as:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o},$$

where ε_o is the permittivity of free space ($8.85\text{x}10^{-12}$ C^2/joule·m).

d.) The above expression is called Gauss's Law. It is a very powerful tool for deriving electric field functions when symmetry is present.

C.) Gauss's Law in Practice--Spherical Symmetry:

1.) Consider a single point charge Q sitting in space. We will use Gauss's Law to determine an expression for the electric field a distance r units from the field-producing charge Q.

2.) Gauss's Law is predicated on the assertion that if a charge is enclosed inside an imaginary surface, the net electric flux through the surface will equal the net charge enclosed within the surface divided by a constant. This assertion is always true, though one has to be careful how the relationship is used. To see this:

a.) Consider an imaginary spherical surface (this is called a Gaussian surface) asymmetrically positioned about a charge Q.

b.) Gauss's Law does work in this situation. Unfortunately, due to the asymmetry, both the angle between E and dS and the magnitude of E vary from point to point over the surface (see Figure 14.4).

In other words, integrating the dot product between E and dS does yield a numerical result that equals Q/ε_o, but the mathematics is so cumbersome that its execution is almost impossible to accomplish.

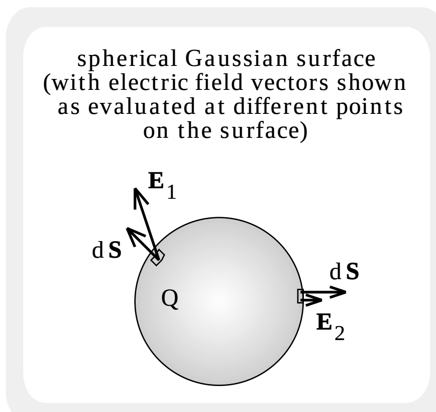


FIGURE 14.4

3.) A more intelligent way to approach this problem is to place the Gaussian surface symmetrically about the charge (see Figure 14.5). Doing so yields a number of useful consequences:

a.) The electric field vector E is the same magnitude at every point on the surface S.

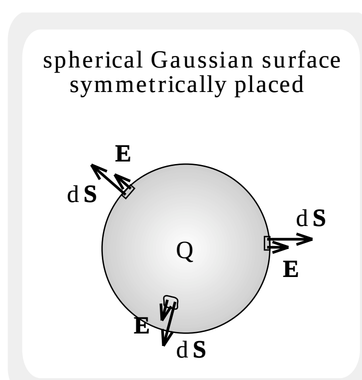


FIGURE 14.5

b.) The vectors E and dS are parallel to one another at every point on S.

4.) With this placement of the Gaussian surface, we can easily use Gauss's Law as follows:

a.) Draw the appropriate Gaussian surface, given the symmetry of the charge configuration (we have already done this above).

b.) Gauss's Law states:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{\text{enclosed}}}{\varepsilon_o}.$$

c.) There are two distinct operations going on in the evaluation of this equation. The right-hand side of the expression requires one to determine the amount of charge enclosed within the Gaussian surface. In our problem, that is easy--it is simply Q.

The left-hand part of the expression requires us to evaluate the integral $\int E \cdot dS$. The evaluation of that integral is done in steps below so as to allow for commentary.

d.) Noting that E and dS are in the same direction, we can write the dot product associated with our integral as:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \int_S E(dS)\cos\theta,$$

where E is the magnitude of E, dS is the magnitude of dS, and θ is the angle between the two vectors. As $\theta = 0$, we can re-write the expression as:

$$\int_S E(dS).$$

Due to the symmetry in the problem, the magnitude of E is the same at every point on the surface. That means we can pull the E term out of the integral, yielding:

$$E\int_S dS.$$

This is great. The integral sums the differential surface areas over the entire surface. As such, it equals the surface area of the entire sphere. For a sphere, this is $4\pi r^2$ where r is the sphere's radius. With this, we can write:

$$E\int_S dS = E(4\pi r^2).$$

e.) Doing the problem as you would on a test:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E(dS)\cos 0^o = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\int_S (dS) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E(4\pi r^2) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2}.$$

This is exactly the expression we derived for the electric field produced by a point charge using Coulomb's Law.

Big Note #1: When dealing with spherical symmetry, the left-hand side of Gauss's equation will always equal $E(4\pi r^2)$ after evaluation. That is, you should ALWAYS be able to write out the left half of Gauss's equation, given the symmetry is spherical.

f.) (Big Note #2 is so large that it has been included as a lettered part of the commentary).

The question arises, "What would have happened if the charge had been negative?" There are a number of ways to approach this. The following is the protocol we will follow:

i.) Assume the electric field E and the differential surface area dS are ALWAYS outward from the surface S. That means the angle between the two vectors will always be zero and the dot product will always be E(dS).

ii.) Always include the sign of the net charge enclosed inside the Gaussian surface when determining $q_{enclosed}$. That means:

1.) If the net charge is positive, your solution for the magnitude of E will be positive and you will know that your assumed direction for the E field was correct (i.e., the field is outward).

2.) If the net charge is negative, your solution for the magnitude of the electric field will be negative. As magnitudes should never be negative, this means you have assumed the wrong direction for the field. Such an observation requires no changes on your part. It simply trumpets the correct direction of the net field (i.e., it is inward, not outward).

D.) More Fun With Spherical Symmetry:

1.) Consider a sphere of outer radius R_2 that has a spherical hole of radius R_1 at its center (see Figure 14.6 to the right). The volume charge density inside the solid part of the sphere is $\rho = ka$, where a is the distance between the sphere's center and a point inside the solid portion of the shell. With that information, what is E(r) for $r < R_1$, for $R_1 < r < R_2$, and for $r > R_2$, where r is an arbitrary distance from the sphere's center to a point in the region of interest?

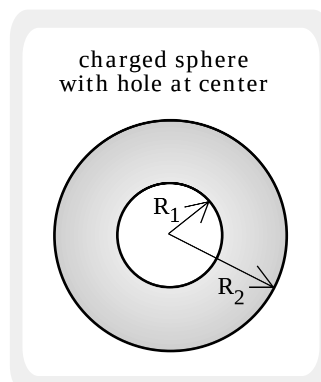


FIGURE 14.6

2.) For $r < R_1$:

a.) Figure 14.7 shows an imaginary Gaussian surface positioned at an arbitrary distance r from the center of the spheres.

b.) Using Gauss's Law, we get:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$= 0$$

$$\Rightarrow E = 0.$$

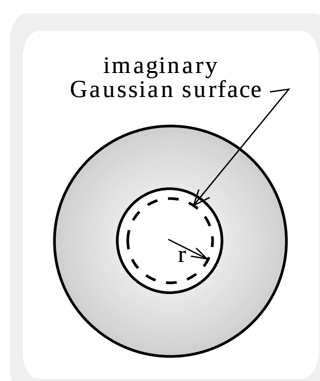


FIGURE 14.7

c.) Does this follow? Certainly. The net charge enclosed within the Gaussian surface is zero. That means the net flux through the Gaussian surface must be zero. The vector dS is not zero, so the only way the integral $\int E(ds)$ can equal zero is if E evaluated on the surface is zero.

Big Note: It is usually about this time that someone says, "What about all the charge outside the Gaussian surface? Why isn't it producing any field at r?"

The answer is, "It is." It just isn't producing any NET field at r. How so? Remember in the chapter on gravitational effects when we dealt with a body inside the earth? We determined that the only mass producing a net gravitational force on such a body is the mass inside the sphere upon which the body sits. Why? Because the gravitational attraction between two point masses is an inverse distance squared function ($F \propto 1/r^2$). With that kind of function, the forces due to all the bits of mass outside the sphere add to zero.

The electric field produced by a point charge is also an inverse-distance-squared function. What that means is that if there is symmetry, the vector sum of the electric fields produced by the charges distributed outside the Gaussian surface will add to zero.

Small Note: Another way to look at the problem outlined in the Big Note above is to remember that Gauss's Law deals with electric flux. A complex, symmetric charge distribution outside a Gaussian surface will produce no net flux through the surface (whatever field lines pass into the volume defined by the surface will sooner or later pass out of the surface). If, then, you are willing to accept Gauss's assumption that the net flux through the surface must be related to charge inside the surface, it naturally follows that FOR SYMMETRIC SITUATIONS, the net electric field generating the flux through the Gaussian surface must be due only to the charge enclosed by the surface.

In short, FOR A SYMMETRIC FIELD, the only charge we ever need to worry about is that enclosed by the Gaussian surface.

3.) For $R_1 < r < R_2$:

a.) Figure 14.8 shows an imaginary Gaussian surface positioned an arbitrary distance r units from the sphere's center.

b.) Before we can use Gauss's Law, we need to determine the charge inside the Gaussian surface. To do so:

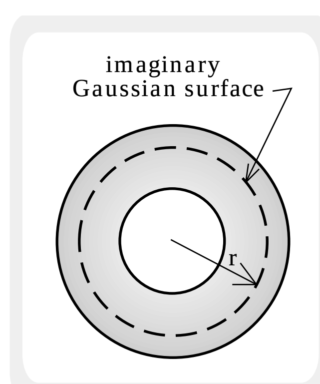


FIGURE 14.8

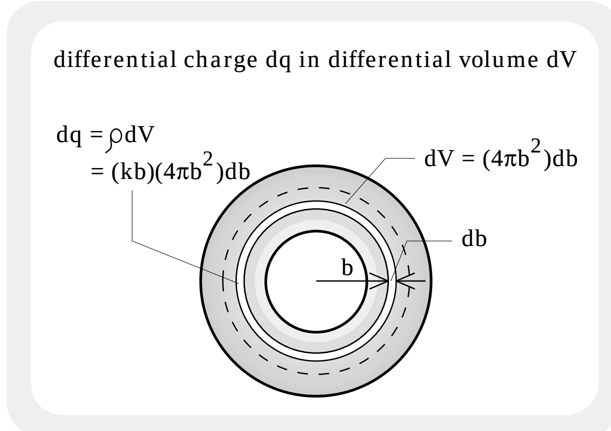


FIGURE 14.9

i.) Define a differential, spherical shell of radius b and thickness db (see Figure 14.9). The differential volume of that shell is:

$$\begin{aligned} dV &= (\text{surface area})(\text{thickness}) \\ &= (4\pi b^2)db. \end{aligned}$$

ii.) The differential charge dq in that volume equals the volume charge density evaluated at b times the differential volume, or:

$$\begin{aligned} dq &= \rho dV \\ &= (kb)[(4\pi b^2)db]. \end{aligned}$$

c.) Gauss's Law yields:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E(dS) = \frac{\int dq}{\varepsilon_o}$$

$$E\int_S (dS) = \frac{\int (\rho)dV}{\varepsilon_o}$$

$$E(4\pi r^2) = \frac{\int_{b=R_1}^{r} (kb)\left[(4\pi b^2)db\right]}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{1}{4\pi r^2}\left(\frac{4\pi k}{\varepsilon_o}\right)\int_{b=R_1}^{r}(b^3)db$$

$$= \frac{k}{r^2\varepsilon_o}\left[\frac{b^4}{4}\right]_{b=R_1}^{r}$$

$$= \frac{k}{r^2\varepsilon_o}\left[\frac{r^4}{4} - \frac{R_1{}^4}{4}\right].$$

4.) For $r > R_2$:

a.) Figure 14.10 shows a Gaussian surface positioned an arbitrary distance r units from the sphere's center.

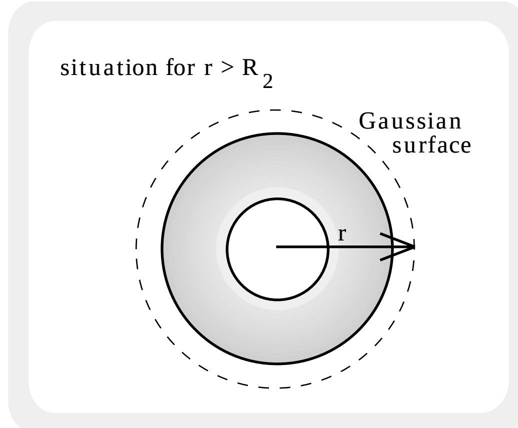


FIGURE 14.10

b.) The left side of Gauss's Law is as usual for spherical symmetry, given that r is the radius of the Gaussian surface.

c.) The right side of Gauss's Law looks very similar to the previous problem (i.e., E(r) for $R_1 < r < R_2$). The only difference is that $q_{enclosed}$ is all the charge in the spherical shell.

i.) Consequence: Gauss's Law will look exactly as it did in the previous problem with the exception that the limits of integration to determine q_{encl} will be from R_1 to R_2 instead of from R_1 to r.

d.) As such, Gauss's Law for this part looks like:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E(dS) = \frac{\int dq}{\varepsilon_o}$$

$$E(4\pi r^2) = \frac{\int_{b=R_1}^{R_2} (kb)\left[(4\pi b^2)db\right]}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{k}{r^2\varepsilon_o}\left[\frac{b^4}{4}\right]_{b=R_1}^{R_2}$$

$$= \frac{k}{r^2\varepsilon_o}\left[\frac{{R_2}^4}{4} - \frac{{R_1}^4}{4}\right].$$

E.) One More Twist--Surface Charge Density:

1.) There is one other kind of charge density function available that hasn't yet been discussed. A surface charge density function σ defines the amount of charge per unit area a structure has on its surface.

a.) Re-consider the thick, spherical shell problem done in Part d. Along with the volume charge density ρ shot through the solid between R_1 and R_2, assume an additional surface charge density $-\sigma$ has been placed on the sphere's outer surface (we are making it negative just to be different--the negative sign has been unembedded to make it clear what kind of charge is present). In that case, determine E(r) for $r < R_1$, for $R_1 < r < R_2$, and for $r > R_2$.

b.) There is to be no difference in analyzing E(r) for $r < R_1$ and for $R_1 < r < R_2$. Why? Because the extra charge has been placed outside the defined Gaussian surfaces for both of these regions.

c.) As the problem is now set up, only the field for $r > R_2$ will be different. Specifically:

$$q_{enclosed} = \int \rho dV - \sigma A,$$

where A is the area of the sphere's outside surface (this will be $4\pi R_2{}^2$).

d.) From looking at the $q_{enclosed}$ expression above, two things should be noted:

i.) Always add all the charge--SIGNS INCLUDED--inside a Gaussian surface.

ii.) You will never encounter a surface charge density function that is not constant. Why? Because a variable density function loses the charge symmetry required for Gauss's Law to work.

The consequence of this surface-charge-density-functions-must-be-constant observation is that we will never have to do anything with differential area dA's.

e.) Just to be complete, Gauss's Law for this problem (assuming $r > R_2$) reads:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{\left[\int \rho dV\right] - \left[\sigma(4\pi R_2^{\ 2})\right]}{\varepsilon_o}.$$

F.) Conductors:

1.) The difference between a conductor and an insulator has to do with the kind of bonding utilized.

a.) Insulators use covalent bonding. Such bonding works by the sharing of outer shell (valence) electrons between neighboring atoms. In insulators, an atom's electrons never stray farther than a few atomic radii from their parent atom.

b.) Conductors use metallic bonding. Such bonding works by the sharing of outer-shell electrons between all other atoms in the structure. In conductors, valence electrons can migrate freely throughout the structure.

2.) Free charge is accelerated when in an electric field. If there is a net, static, electric field permeating a conductor, the valence electrons within the conductor will move until they position themselves so as to eliminate the field (i.e., they can't stop until the net field is zero). In short, FOR A STATIC CHARGE SITUATION, THE ELECTRIC FIELD INSIDE A CONDUCTOR WILL ALWAYS BE ZERO.

3.) With that in mind, consider the following situation: Figure 14.11 shows a point charge -Q placed at the center of a thick, hollow, conducting spherical shell of inner radius R_1 and outer radius R_2 (again, we are using a negative charge to be different). The conductor is electrically neutral (that is, there is no extra charge placed on the sphere--there are as many electrons as protons in the structure).

What is the E(r) for $r < R_1$ and for $R_1 < r < R_2$?

a.) For $r < R_1$: The charge enclosed is equal to -Q. Using Gauss's Law for spherical symmetry in very brief form yields:

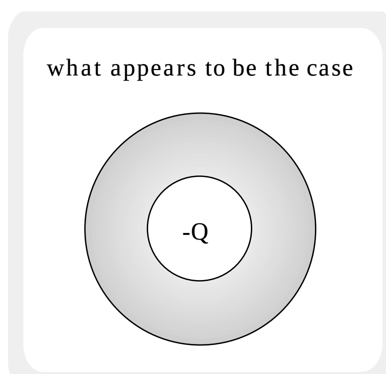


FIGURE 14.11

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow E = \frac{-Q}{4\pi\varepsilon_o r^2},$$ where the negative sign means the field is inward.

b.) For $R_1 < r < R_2$: The temptation is to assume the same solution holds for this region as held for $r < R_1$. After all, a Gaussian surface extending into the region between R_1 and R_2 appears to have -Q's worth of charge inside it (again, see Figure 14.11).

If that is the case, the net charge inside the Gaussian surface should produce an electric field in the region. The problem? The sphere is a CONDUCTOR. There can't be an electric field in the region.

How do we resolve this seeming paradox?

c.) When the -Q charge is first introduced at the sphere's center, it briefly generates a net electric field inside the metallic sphere. Electrons in the sphere respond to the field by re-distributing themselves so as to eliminate the field (this takes only a few millionths of a second to do).

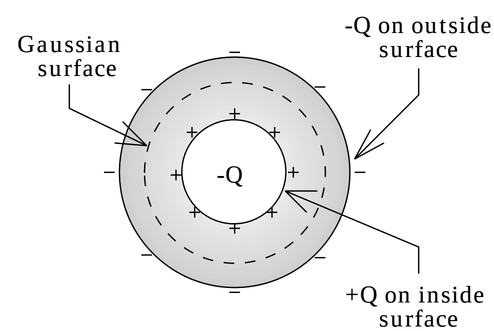
How did they re-distribute?

-Q's worth of valence electrons within the sphere migrate away from the -Q at the sphere's center, leaving +Q's worth of charge on the sphere's inside surface. In doing so, the amount of charge inside the Gaussian surface between R_1 and R_2 becomes zero.

Where does the -Q's worth of valence charges go? They move to the outside surface of the sphere (see Figure 14.12).

d.) <u>Bottom Line</u>: AGAIN, THERE CAN BE NO STATIC ELECTRIC FIELD INSIDE A CONDUCTOR. If a field is momentarily induced, the charges within the conductor will re-arrange themselves so as to eliminate it. As such, a Gaussian surface located inside

What is really the case?
Electrons in the conductor are repulsed to the outside surface by the presence of -Q at the center. That leaves +Q on the sphere's inner surface.



Note: The net charge inside the Gaussian surface between R_1 and R_2 is ZERO.

FIGURE 14.12

a conducting volume will ALWAYS have NO NET CHARGE within it.

G.) Gauss's Law in Practice--Cylindrical Symmetry:

1.) Consider a very long wire with a constant, linear charge density λ on it (a linear charge density function defines the amount of charge per unit length on a structure). As long as our attention is focused in close around the central section of the wire, we can ignore edge effects and assume the electric field is radially outward (or we could do what most books do and make the highly unrealistic assumption that the wire is infinitely long).

Use Gauss's Law to derive an expression for E(r), where r is an arbitrary distance radially out from the wire (i.e., perpendicularly out).

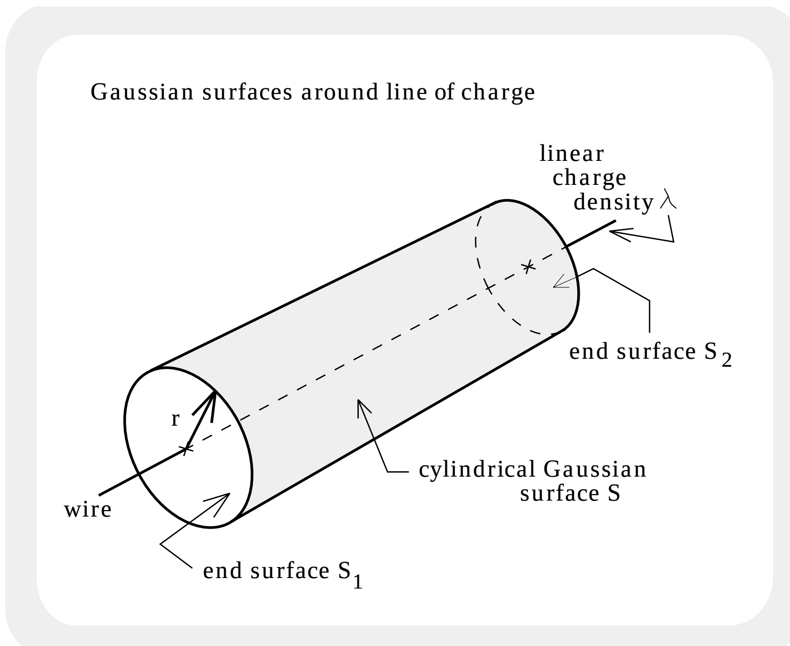


FIGURE 14.13

2.) Again, assuming we are not near the ends of the wire, the symmetry here is cylindrical. How so? Examine Figure 14.13. It shows an imaginary, cylindrical Gaussian surface positioned at an arbitrary distance r from the wire.

3.) This Gaussian surface is actually three surfaces in one--two flat, circular surfaces at each end and the cylinder itself.

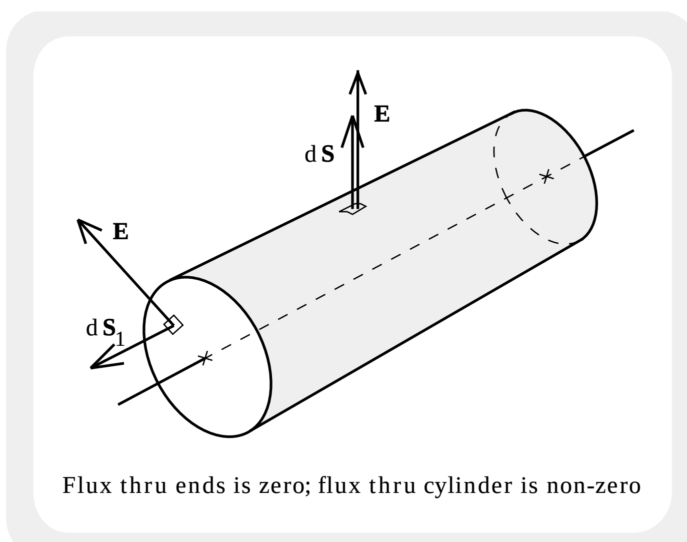


FIGURE 14.14

a.) Figure 14.14 defines E and dS for both the cylindrical part of the structure as well as for one of the flat end-pieces.

b.) As E and dS_1 on the end are at a right angle to each other, their dot product equals zero. The same is true at the other end.

Note 1: If we were not dealing with the central section of the wire, the electric field would not be radially outward, there would be flux through the end-pieces, and we would not be able to use Gauss's Law to determine the electric field function we want. Also, as the wire is not infinitely long, the electric field expression we derive here will only be good near the wire (versus being good far from the wire . . . like at infinity).

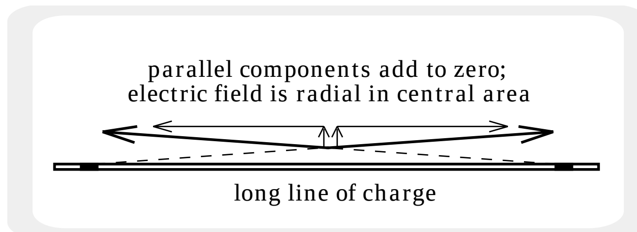


FIGURE 14.15

Note 2: Still don't believe the field will be radially outward? Consider the following: For any point near the center of the wire, there will be electric fields generated by bits of charge all along the wire. Take two mirror charges (i.e., differential charges on opposite sides and approximately the same distance from the central section). The components of the electric field parallel to the wire will add to zero leaving only the components radially out from the wire (see Figure 14.15).

Note 3: Does that mean the electric field is caused by charge carriers all along the wire? Yes!

Does that mean that the charge outside the Gaussian surface has to be taken into account when using Gauss's Law?

NO! NO! NO! Gauss's Law doesn't say that the electric field is generated by the charge inside a Gaussian surface, it says that the FLUX THROUGH THE GAUSSIAN SURFACE is related to the charge enclosed (the net flux from charges outside the surface must be zero). We are able to deduce the electric field associated with that flux, whatever the accumulated charge has made that field to be, only because there is a mathematical relationship between an electric flux and its electric field.

4.) As the electric flux through the ends is zero, the only flux present is through the cylindrical part of the structure. With that in mind, we can write Gauss's Law as:

$$\int_{\text{cyl.Surf}} \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enc}}{\varepsilon_o}.$$

5.) Examining the integral:

a.) As the angle between E and dS is zero on the cylinder, the dot product inside the integral is simply E(dS).

b.) Due to symmetry, the magnitude of E is the same at every point on the cylinder's surface. That means we can pull the E term outside the integral just as we did when we dealt with spherical symmetry.

c.) The sum of the differential surface areas over the entire cylindrical surface (i.e., $\int dS$) equals the surface area of the cylinder. The surface area of a cylindrical shell is:

$$(\text{circumference})(\text{length}) = (2\pi r)L,$$

where r is the radius and L is the length of the Gaussian surface.

d.) <u>Bottom Line</u>: For cylindrical symmetry, the left-hand side of Gauss's Law will ALWAYS equal:

$$E(2\pi rL).$$

6.) The right-hand side of Gauss's Law requires a determination of the amount of charge inside the Gaussian surface.

a.) In this case, the charge per unit length is λ.

b.) The wire inside the Gaussian surface has a length L, so the charge inside the Gaussian surface must be:

$$\begin{aligned} q_{encl} &= (\lambda \text{ C/m})(L \text{ meters}) \\ &= \lambda L. \end{aligned}$$

7.) Writing this out formally (i.e., in the way you might do on a test), we get:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enc}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E(dS)\cos 0^o = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E\int_S dS = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi r L) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_o r}.$$

Note: The L terms canceled. This is good! It would have made no sense to have concluded that our electric field was dependent upon the length of the imaginary Gaussian surface used to generate the electric field expression.

H.) Deeper and Deeper into Cylindrical Symmetry:

1.) Consider a long rod of radius r_1 with a volume charge density of kr shot through it. Coaxially, a cylindrical, conducting shell of inside radius r_2 and outside radius r_3 is placed around the rod (see Figure 14.16). On the shell is a constant surface charge density $-\sigma$. Determine E(r) for:

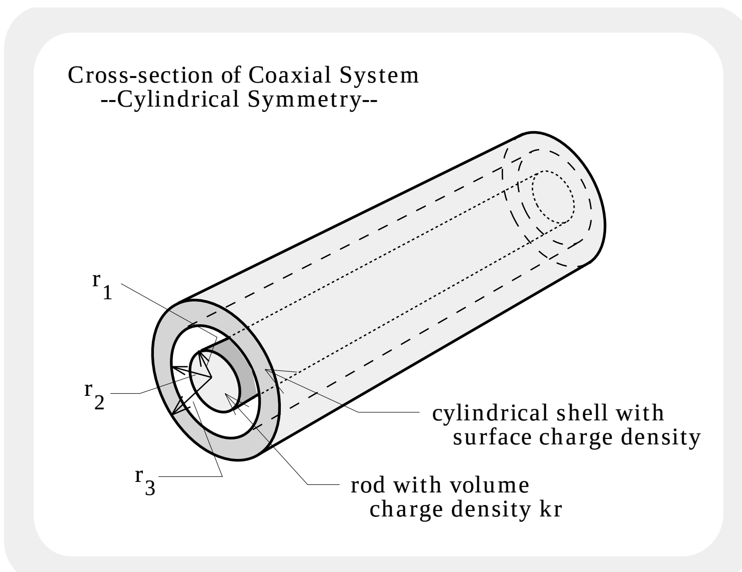


FIGURE 14.16

a.) $r < r_1$;

b.) $r_1 < r < r_2$;

c.) $r_2 < r < r_3$;

d.) $r_3 < r$.

Note: This is typically the way test problems are stated.

2.) For $r < r_1$:

a.) The left-hand side of Gauss's Law is always the same for cylindrical symmetry. That is, the integral ends up equaling $E(2\pi rL)$. Again, we are assuming that both E and dS are outward, understanding that if this is not true the sign of the charge will correct the oversight.

b.) The right-hand side requires the use of a volume charge density function to determine the charge enclosed within the cylindrical Gaussian surface.

Let a be the distance from the central axis to a differentially thin, cylindrical shell of length L (i.e., the length of the Gaussian surface) and differential thickness da (to see all of this, Figure 14.17 shows a blown-up sketch of the rod by itself).

The volume is:

$$dV = (\text{surface area of the cylinder})(\text{thickness})$$
$$= (2\pi aL)da.$$

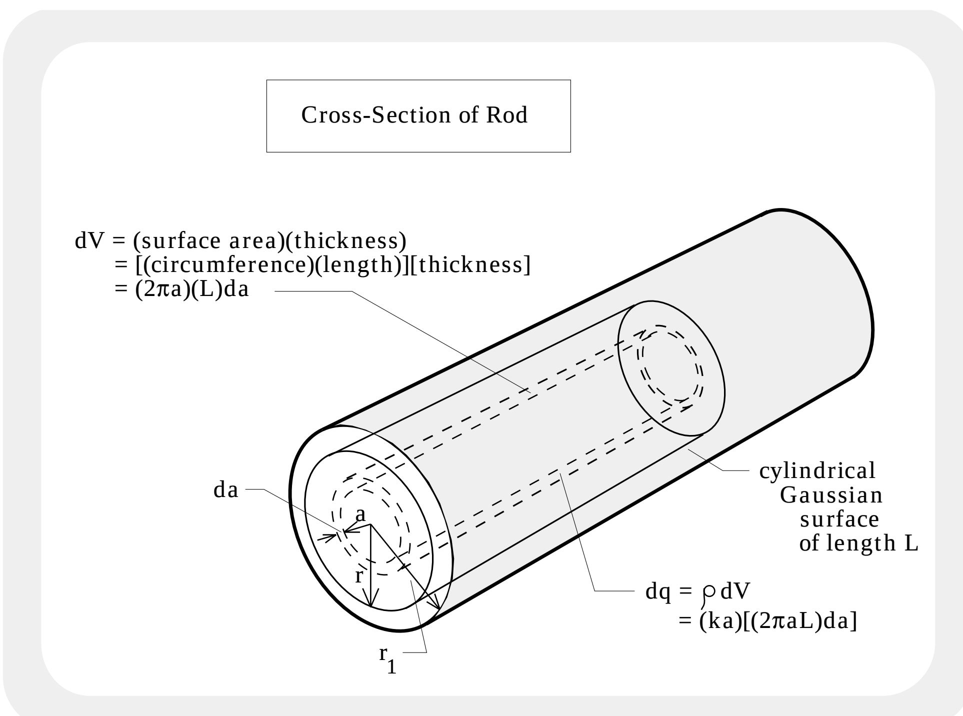


FIGURE 14.17

while the differential charge enclosed in that differential volume is:

$$\begin{aligned} dq &= \rho dV \\ &= (ka)[(2\pi aL)da]. \end{aligned}$$

c.) Using this information, and noting that the charge within the Gaussian surface is distributed between a = 0 and a = r, we can write:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \qquad \int_S E(dS) = \frac{\int \rho dV}{\varepsilon_o}$$

$$E\int_S (dS) = \frac{\int (ka)[(2\pi aL)da]}{\varepsilon_o}$$

$$E(2\pi rL) = \frac{2\pi kL\int_{a=0}^{r} a^2 da}{\varepsilon_o}$$

$$\Rightarrow E = \frac{k}{\varepsilon_o r}\left[\frac{a^3}{3}\right]_{a=0}^{r}$$

$$= \frac{kr^2}{3\varepsilon_o}.$$

3.) For $r_1 < r < r_2$:

a.) In the region between r_1 and r_2, the charge enclosed inside an imaginary, cylindrical Gaussian surface of radius r will be the total charge inside the rod. Assuming the radius of the Gaussian surface is r, the only difference in this problem and the problem in Part 2 will be the limits of integration.

i.) How so? In Part 2 we integrated from a = 0 to a = r. Why? Because we needed the charge inside the Gaussian surface which was, in turn, inside the rod. As such, our summing had to terminate inside the rod.

ii.) In this case, our Gaussian surface is outside the rod. That means we must terminate our integration where the charge ceases to exist (i.e., at r_1), not at r.

b.) In a somewhat truncated form, our derivation for this section follows as:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{\int (\rho)dV}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rL) = \frac{\int_{a=0}^{r_1} (ka)(2\pi aL)da}{\varepsilon_o}$$

$$\Rightarrow \quad E(r) = \frac{k}{\varepsilon_o r} \int_{a=0}^{r_1} a^2 da$$

$$= \frac{k}{\varepsilon_o r}\left(\frac{r_1^{\,3}}{3}\right).$$

4.) For $r_2 < r < r_3$:

a.) The short answer: The field must be zero as the Gaussian cylinder resides inside a conductor.

b.) The long answer: Positive charge inside the rod attracts electrons inside the conductor to the conductor's inner wall. How much charge? An amount equal to the amount of positive charge on the rod.

Consequence: The net charge inside the Gaussian surface is zero and the net field evaluated on the surface of the Gaussian cylinder must also be zero.

5.) For $r_3 < r$:

a.) Figure 14.18 shows the system and the Gaussian surface that is appropriate to this problem.

b.) The charge enclosed in the Gaussian surface comes from the positive charge distributed inside the rod (i.e., from a

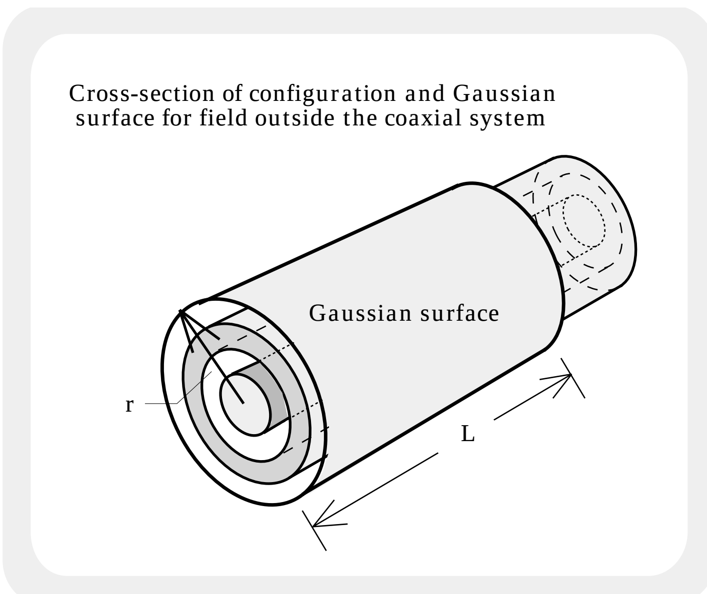


FIGURE 14.18

= 0 to a = r_1) and the net negative charge that was initially placed on the conductor (the induced charge on the inside and outside surfaces of the conductor will add to zero).

c.) Using Gauss's Law, we can write:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \qquad E(2\pi rL) = \frac{\int_{a=0}^{r_1} (ka)(2\pi aL)da + (-\sigma)A}{\varepsilon_o}.$$

d.) The integral portion of $q_{enclosed}$ is the same as in Part 3.

e.) A is defined as the area of that portion of the outside surface of the cylindrical shell found inside the Gaussian surface. Mathematically, it is A = (circumference)(length) = $(2\pi r_3)$L.

f.) Doing the integral, substituting in for A, and dividing out the $2\pi rL$ terms leaves:

$$E = \frac{k\frac{r_1^{\,3}}{3} - \sigma r_3}{\varepsilon_o r}.$$

I.) Insulating Sheets:

1.) A thick, rectangular, insulating solid is permeated with a uniform charge density throughout its volume (see Figure 14.19).

a.) We could assign a volume charge density to the object and go from there.

b.) An alternative approach would be to define a surface area charge density function (i.e., σ) for the structure. That is, we could associate the amount of charge in a given volume with the surface area

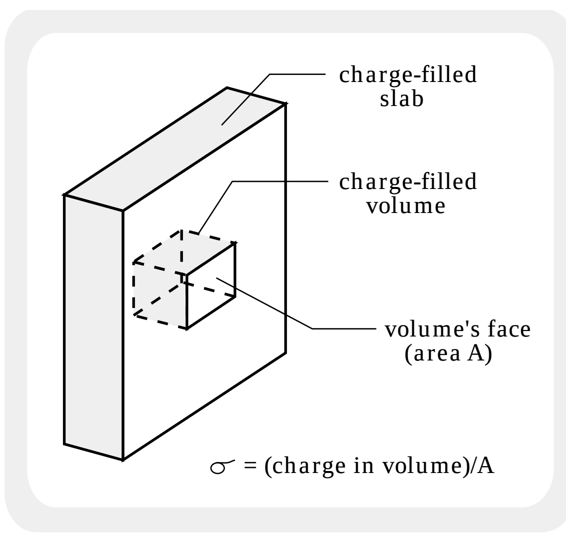


FIGURE 14.19

of the volume's face (again, see Figure 14.19). Mathematically, this would be:

$$\sigma = \text{(charge inside volume)/(face area of volume)}.$$

c.) The rationale for creating such a function is simple. With it we can use Gauss's Law to derive an expression for the electric field just outside any uniform, charge-filled insulator.

How so? Continue on!

2.) Consider a very thin, flat sheet of charge with a surface charge density σ coulombs per square meter associated with it. What is the electric field close to the surface and around the central section of the sheet?

3.) Figure 14.20a shows the set-up along with a cubic Gaussian surface, where A is the face area of the surface.

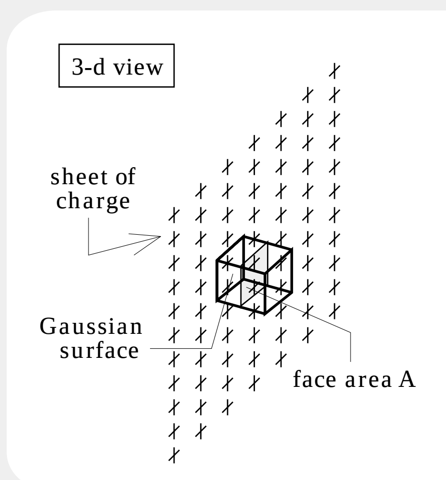


FIGURE 14.20a

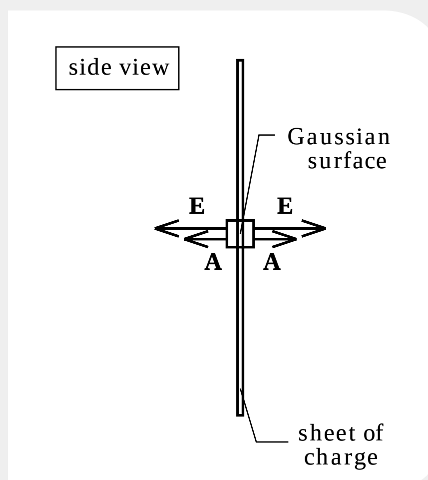


FIGURE 14.20b

Note: As long as the sides of the Gaussian surface are either parallel or perpendicular to the sheet itself, the shape of the Gaussian surface is inconsequential. That is, we are using a cube here; it could as well have been a cylindrical plug.

4.) Ignoring edge effects (we are dealing with the area in the central section of the sheet) and assuming we are looking for the field very near the sheet, the electric field will be perpendicularly out from the sheet. That means:

a.) The flux through each side-face (that is, each face perpendicular to the plane of the sheet) is zero.

b.) Assuming the cube is positioned so that the parallel faces are symmetrically placed about the sheet:

i.) The electric field passing through both faces will have the same magnitude (note the direction through both faces is outward).

ii.) Examining the side-view of the system in Figure 14.20b, the dot product used to determine the flux through <u>each</u> of the two parallel faces will be positive and equal to EA.

5.) Gauss's Law yields:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad EA + EA = \frac{\sigma A}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_o}.$$

6.) This expression is always true whenever we want the electric field very near the central section of a sheet of charge or a charge-filled insulating slab for which we have defined a surface charge density function.

7.) Does it make sense for this function to be independent of the distance from the surface?

Yes!
How so?
Follow along:

a.) When the point of interest (call it Point A) is very near:

i.) Charges on the surface close to Point A will produce large electric fields perpendicularly out from the surface. They will also produce small electric fields parallel to the surface, but those will add to zero when all the near-charge fields are vectorially added.

ii.) Charges on the periphery will produce smaller net electric fields. Those fields will have large components parallel to the surface which will add to zero when vectorially summed. They will also have smaller components perpendicularly out from the surface. Figure 14.21 shows all.

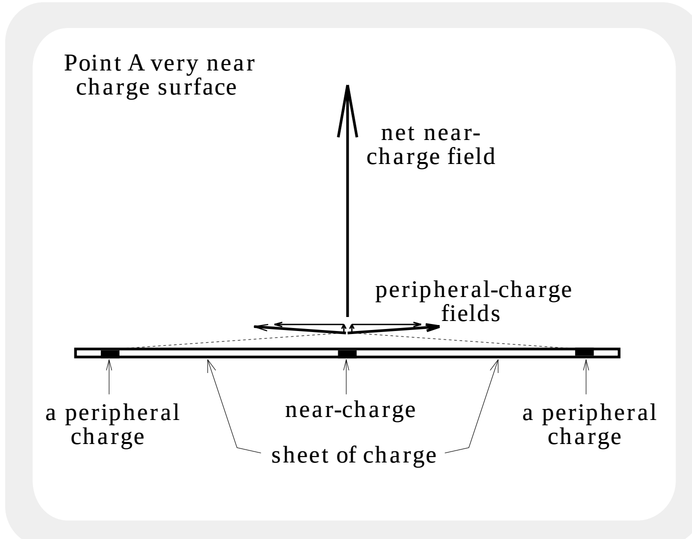


FIGURE 14.21

iii.) When Point A moves out away from the surface, the amount of field produced by the near-charge fields will diminish. The peripheral-charge field will also diminish, but the perpendicular component of those charges will become proportionally larger (see Figure 14.22). The net effect is that the net electric field doesn't change.

iv.) This works only when relatively close to the surface. How close? Good question. If you have nothing better to do, try to determine how far out from a 1 meter radius, circular surface (this geometry will be easier to deal with) you can get and still have the actual electric field value be within 5% of $\sigma/2\varepsilon_o$.

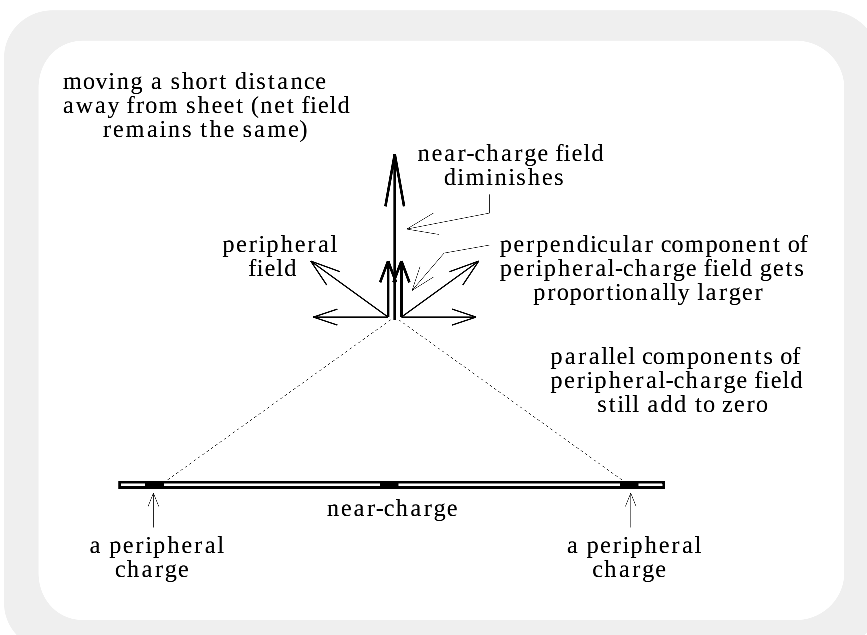


FIGURE 14.22

Note: If this were really important, I'd have an answer. It isn't, so instead of an answer you get a problem at the back of the chapter.

J.) Conducting Sheets:

1.) A thick, rectangular, conducting solid has charge uniformly distributed over its entire surface (assume the surface charge density is σ). What is the electric field close to the sheet and around its central region?

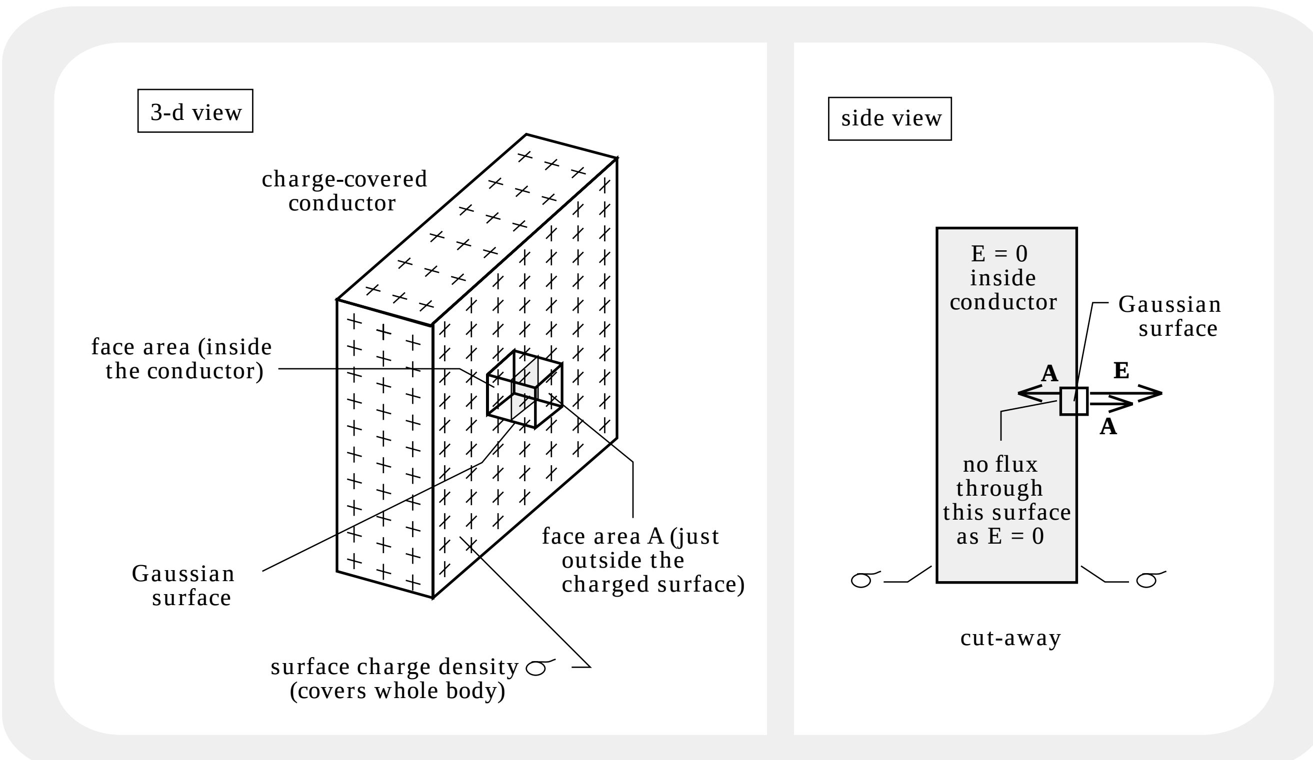


FIGURE 14.23a FIGURE 14.23b

2.) There are two ways to do this. The first is embodied in Figures 14.23a and 14.23b. A Gaussian plug (as before, it happens to be a cube, though it could have been a cylindrical plug) is placed so that one face is inside the conductor.

a.) As the electric field inside a conductor is zero, the flux through the left face will be zero. That leaves flux passing only through the right face.

b.) Using Gauss's Law, we get:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad EA = \frac{\sigma A}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_o}.$$

c.) Notice that this expression is very similar to the one derived for a thin sheet of charge (or an insulating slab with a surface charge density function associated with it).

3.) An alternative approach would be to extend the Gaussian plug so that the end faces are the same distance from their respective outer surfaces (see Figure 14.24).

a.) In that case, there will be a flux through both end faces, but there will also be twice as much charge within the Gaussian surface (there is σ worth of charge per unit area on all outer surfaces of the structure).

b.) As such, Gauss's Law becomes:

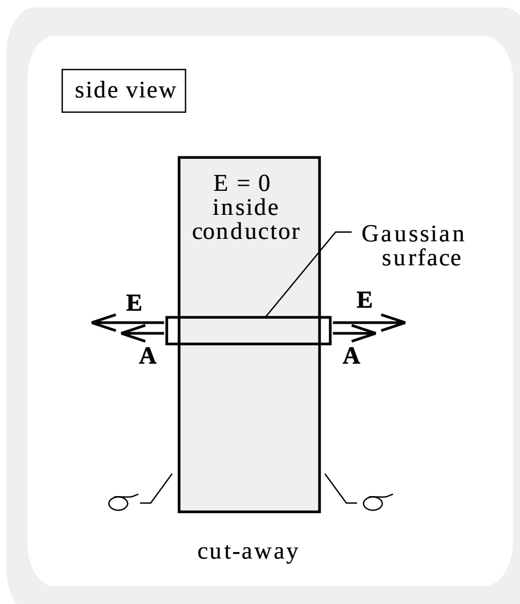


FIGURE 14.24

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad EA + EA = \frac{\sigma A + \sigma A}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_o}.$$

QUESTIONS

14.1) A ski cap shown to the right in Figure I has an open hole whose area is A_o. The total surface area of the cap is $5A_o$. If the cap's hole is positioned so that it is in the x-z plane and an electric field in the y-direction permeates the region in which the cap resides, what is the net electric flux through the cap's surface?

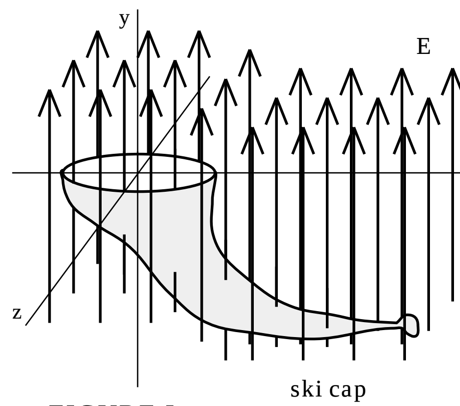


FIGURE I

14.2) A spherical insulator of radius R has Q's worth of charge uniformly distributed throughout it. Determine the electric field E(r) for:

a.) $r < R$, and;

b.) $r > R$.

14.3) A spherical, conducting shell has an inside radius R_1 and outside radius R_2. Q's worth of charge is somehow levitated at the shell's center while a charged rod is used to initially place 2Q's worth of charge on the shell's inner surface:

a.) After a short time, how much charge will end up on the inner surface of the shell?

b.) How much charge will end up on the outer surface of the shell?

c.) Derive an expression for E(r), where $r < R_1$;

d.) Derive an expression for E(r), where $R_2 > r > R_1$; and

e.) Derive an expression for E(r), where $r > R_2$.

f.) Out of curiosity, what is the surface charge density on the outer surface of the shell?

14.4) A spherical shell has an inside radius R_1, an outside radius R_2, and a volume charge density defined by $(C/r^2)e^{kr}$ (note that r is the distance from the shell's center to a volume of interest, k is a constant equal to one and having the appropriate units, and C is also a constant having the appropriate units). Layered over this shell is a conducting shell whose inside radius is R_2, whose outside radius is R_3, and upon which has been placed a constant surface charge density σ.

a.) What are k's units; what are C's units?

b.) Accurately plot the Electric Field versus Position graph for this charge configuration. Note that this is not a gee whiz question. Use Gauss's Law to determine the electric field expression for each region, then graph it.

For the sake of simplicity, assume that $C = 10^{-12}$, $\sigma = 10^{-12}$ coulombs per square meter, $R_1 = 1$ meter, $R_2 = 2$ meters, and $R_3 = 3$ meters. Use these values only after you have derived your general expressions.

Note: DO NOT TAKE A LOT OF TIME ON THE GRAPH. WHAT IS IMPORTANT HERE IS USING GAUSS'S LAW TO DERIVE THE GENERAL ELECTRIC FIELD EXPRESSIONS FOR EACH REGION.

c.) Looking at the graph, would you say that electric field functions are continuous?

14.5) A thick-skinned pipe of inside radius r_1 and outside radius r_2 has a surface charge density σ_1 on its inside surface and a surface charge density σ_2 on its outside surface ($\sigma_1 \neq \sigma_2$, and the charge associated with σ_2 is negative). A second thick-skinned pipe surrounds the first pipe coaxially. Its inside radius is r_3, its outside radius is r_4, and it has a volume charge density $\rho = (C/r)e^{kr}$ shot through its volume. Figure II shows the layout.

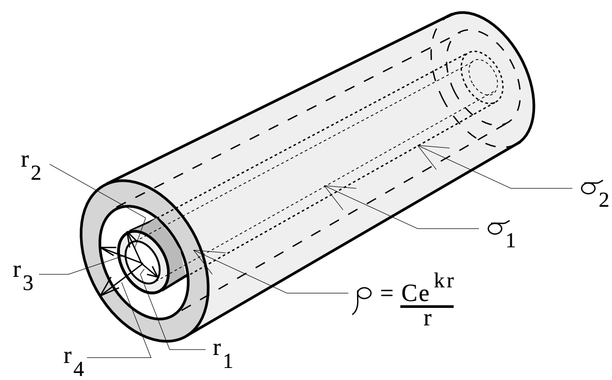


FIGURE II

a.) Given the information provided, can you tell for sure whether either pipe is a conductor or an insulator?

b.) Accurately plot the Electric Field versus Position graph for this charge configuration.

For the sake of simplicity, assume that $C = 10^{-12}$ coulombs per square meter, $\sigma_1 = 10^{-12}$ coulombs per square meter, $\sigma_2 = -2\times10^{-12}$ coulombs per square meter, $r_1 = 1$ meter, $r_2 = 2$ meters, $r_3 = 3$ meters, and $r_4 = 4$ meters. Use these values only after you have derived your expressions.

Note: As in Problem 14.4, do not take a lot of time on the graph.

14.6) A very long cylindrical rod of radius R has an unknown volume charge density shot through it. The electric field a distance r units from the rod's central axis is found to be $E = kR^6/(6\varepsilon_o r)$, where k is a constant and r > R. Use Gauss's Law and your head to determine TWO different volume charge density functions that could produce this field.

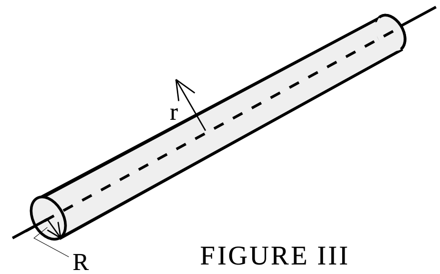


FIGURE III

14.7) A thick, insulating disk of radius R = 1 meter and thickness t = .02 meters has Q's worth of charge uniformly distributed throughout its volume.

a.) Use Gauss's Law and the appropriate volume charge density function to derive an expression for the electric field very close to the disk's surface and along the disk's central axis.

b.) Determine an equivalent surface charge density function for this disk.

c.) (This is more of a review question than anything else.) Using the surface charge density function, treat the disk like a sheet of charge. Use Gauss's Law to determine the electric field down the central axis and very close to the disk's surface. As this field is accurate only when near the disk, call it the near-point field.

d.) Using the old-fashioned definition approach to determine the electric field down the disk's axis (i.e., define a differential point charge on the sheet of charge, determine the field due to that point charge, then integrate to determine the net field due to all the charge). Call this the general, axial field.

e.) The near-point field is an approximation that is good only when near the disk. The general, axial field is good at any point along the axis. How far from the disk can one get and still have the electric field value determined using the near-point field expression be within 5% of the electric field value determined using the general, axial field expression?